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THE OSCILLATIONS OF AN OSCILLATOR NEAR THE INTERFACE BETWEEN TWO LIQUIDS*

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The normal modes of a oscillator (an elastically attached solid) near the interface between two stable stratified liquids are studied. The liquids are assumed to be ideal and incompressible. The motion of the body causes internal waves to radiate along the interface. Numerical-analytical methods are developed to investigate small oscillations of the oscillator and the liquid, in a selfconsistent hydrodynamical model. Qualitative observations are made concerning the damping of the oscillations due to the dispersive properties of the medium, as well as the excitation and propagation of the internal waves. A study is made of the amplitude-frequency characteristic (AFC) of the oscillator incorporating the reaction of the waves; the AFC determines such global properties of the oscillating system as the selectivity, frequency variation, etc.

1. Initial assumptions and statement of the problem. An investigation will be made of small vertical oscillations of the mechanical system represented in Fig.1. It is assumed that a convex solid C with a fairly smooth surface Σ is situated beneath the interface between two ideal incompressible liquids. To fix our ideas, we shall assume that the body is a circular cylinder, whose generatrix remains parallel to the horizon throughout the motion. Let a be the radius of the cylinder and b the length of the generatrix, $b \gg a$, h the depth of immersion, $h \gg a$, ρ_1 the density of the upper liquid layer, $\rho_1 \geqslant 0$, H the thickness of the layer, $0 < H < \infty$, and ρ_2 the density of the infinitely deep lower liquid, $\rho_2 > \rho_1$. The liquids are situated in a uniform gravitational field of acceleration g.

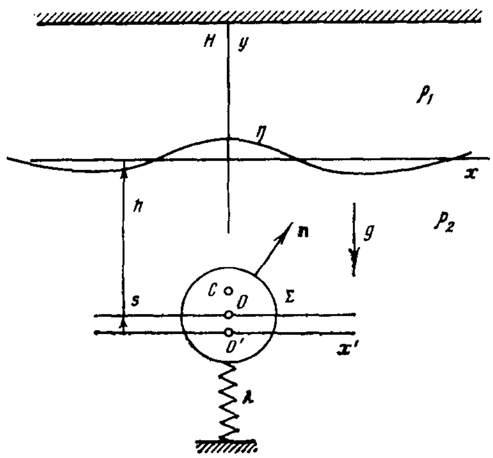

Fig.1

It is assumed that forward motion of the axis O of the cylinder C takes place in a vertical plane perpendicular to the plane of the diagram (xy), under the effect of gravity, the reaction forces exerted by the liquid and a linear restoring force. As a mechanical example of the force restoring the system to some stable equilibrium position we consider a linear elastic member (spring) with compliance λ (Fig.1). Note that since $b \gg a$ we may assume that the motions of the liquid are approximately planar.

The mechanical problem may be formulated as follows. At the start of the motion $t = 0$ the body (its centre O or centre of mass C) is displaced vertically from its equilibrium position O' through a small distance s^0, $|s^0| \ll a$. It is then released at zero initial velocity, and at $t > 0$ begins to oscillate. At the starting time the liquids are also at

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rest, with the interface a horizontal surface, i.e., $y = \eta(x, 0) = 0$, where $y = \eta(x, t)$ is the elevation of the interface. The potential energy of the body is used up in generating waves spreading out from the body, and the oscillations are damped. The problem is to determine the subsequent motion of the oscillator $s(t)$ with allowance for the waves in the surrounding inhomogeneous liquid and the form of the interface $\eta(x, t)$. The motion of the system is analysed using the linear theory of waves motions in a liquid /1/.

There is a voluminous literature on the oscillatory motions of a body floating on the free surface of a homogeneous liquid (/1-5/ et al.). An example is the study in /3/ of the damping of the oscillations of a "Froude buoy". We must mention that, as a rule, authors confine themselves to analysing the asymptotic behaviour of the oscillations when the time increases without limit. The papers cited essentially duplicate the work of Sretenskii /1/. It has been pointed out /4/ that "the motion of the /floating/ body can be found accurately, although little can be said about the wave motion in the fluid". A detailed study has been made /5, 6/ of the oscillations of a thin solid on the interface between a two-layered liquid.

In the present paper we present a complete solution of the problem of an elastically attached cylinder oscillating beneath the interface between two liquids and investigate the wave motions of the liquids. The dynamics of the system depend essentially on three additional parameters: the depth of the upper liquid layer, the quotient of the densities of the liquids and the frequency of the normal modes of the unimmersed oscillator (not reacting with the liquid). This situation greatly enriches the dynamics of the oscillating system compared with the case of a body floating on a free surface or interface between liquids /1-6/. The transient oscillations of an oscillator interacting with an inhomogeneous liquid have not been considered before.

We shall assume henceforth that the displacements $s(t)$ of the body are small, and that the distance of the equilibrium position O' (and the body C) from the interface is sufficiently large. The disturbances of the liquids will therefore be small, enabling us to use the linearized equations of hydrodynamics and approximate conditions of flow around a cylinder. Let $\Phi_{1,2} = \Phi_{1,2}(x, y, t)$ denote the potentials of the velocities for the upper and lower liquids, respectively, in the regions not occupied by the body. Since the liquids are incompressible, they satisfy the Laplace equation in these regions $\forall t \in [0, T]$, $T < \infty$:

$$\begin{gathered} \Delta\Phi_1 = 0, \ |x| < \infty, \ 0 < y < H \\ \Delta\Phi_2 = 0, \ |x| < \infty, \ y < 0, \ (x, y) \overline{\in} C \end{gathered} \tag{1.1}$$

At the solid surfaces the liquids must satisfy boundary conditions of impermeability, $\forall t$:

$$\partial\Phi_1/\partial y\,|_{y=H} = 0, \ \partial\Phi_2/\partial \mathbf{n}\,|_\Sigma = -cn_y \quad (c = s^{\cdot}) \tag{1.2}$$

Here Σ is a circle, $c = c(t)$ is the vertical velocity of the centre of inertia of the cylinder, and n_y is the projection on the y-axis of the normal vector $\mathbf{n}$ to the cylinder surface Σ. Within a finite time, the disturbances in the liquid and along the interface will travel a finite distance; hence the following conditions hold at infinity $\forall t \in [0, T]$:

$$\Phi_2\,|_{y\to-\infty} \to 0, \ \Phi_{1,2}\,|_{|x|\to\infty} \to 0 \tag{1.3}$$

Besides conditions (1.2), (1.3), boundary conditions are also imposed on the liquid interface $y = \eta(x, t)$, expressing the fact that the pressures and normal velocities are equal. In the linear approximation, these conditions are

$$\begin{gathered} \rho_1(\partial\Phi_1/\partial t - g\eta)_{y=0} = \rho_2(\partial\Phi_2/\partial t - g\eta)_{y=0} \\ \partial\eta/\partial t = -\partial\Phi_1/\partial y\,|_{y=0} = -\partial\Phi_2/\partial y\,|_{y=0} \\ (= -{}^1/_2\,\partial(\Phi_1 + \Phi_2)/\partial y\,|_{y=0}) \\ |x| < \infty, \ t \in [0, T], \ t < \infty \end{gathered} \tag{1.4}$$

In order to eliminate the unknown function η, we differentiate the first equality of (1.4) with respect to t and use the second to eliminate $\partial\eta/\partial t$. This gives a system of conditions involving only the functions $\Phi_{1,2}$:

$$\begin{gathered} [\rho_2\partial^2\Phi_2/\partial t^2 - \rho_1\partial^2\Phi_1/\partial t^2 + g(\rho_2 - \rho_1)\,\partial\Phi_{1,2}/\partial y]_{y=0} = 0 \\ \partial\Phi_1/\partial y\,|_{y=0} = \partial\Phi_2/\partial y\,|_{y=0} \ (= {}^1/_2\,\partial(\Phi_1 + \Phi_2)/\partial y\,|_{y=0}) \end{gathered} \tag{1.5}$$

Together with the above boundary conditions (1.2)-(1.5), we need initial conditions for the unknown functions $\Phi_{1,2}, \eta$. In view of our assumptions as listed above, these conditions are trivial:

$$\Phi_{1,2}(x, y, 0) = 0, \ \eta(x, 0) = 0 \tag{1.6}$$

The approach outlined below for solving this problem includes the following steps: a) solution of the external problem of hydrodynamics on the assumption that the motion of the body is known, as formulated in Sect.1 (Sect.2); b) derivation of the selfconsistent equation of motion and formulation of a closed Cauchy problem for the body (Sect.3); c) study of the frequency characteristics of the oscillations, in order to determine the global properties of the motion (Sect.4); d) numerical-analytical solution $s(t)$ of the integrodifferential Cauchy problem (IDCP) for the body and construction of the interface.

2. ***Solution of the hydrodynamic problem given the motion of the cylinder.*** The major difficulty in solving the external hydrodynamic problem is due to the need to observe the boundary conditions on the different components of the boundary, which is disconnected: on the surface Σ of the cylinder and the upper solid wall at $y = H$ (1.2), and also on the interface between the liquids (1.4) or (1.5) (see Fig.1). The conditions on the surface Σ of the cylinder can be satisfied by using a modification of Havelock's method /1/.

We confine ourselves to the dipole approximation: the motion of the liquid near C will be described in terms of a dipole potential placed at the point $(0, -h)$. The required velocity potentials $\Phi_{1,2}$ may be written in the form /1/

$$\begin{gathered}\Phi_1 = \varphi_1(x, y, t), \ |x| < \infty, \ H \geqslant y > 0 \\ \Phi_2 = a^2c(t)\,z(x, y) + \varphi_2(x, y, t), \ |x| < \infty, \ y < 0, \ (x, y) \overline{\in} C \\ z(x, y) \equiv (y+h)[x^2 + (y+h)^2]^{-1} - (y-h)[x^2+(y-h)^2]^{-1}\end{gathered} \tag{2.1}$$

where $\varphi_{1,2}$ are the velocity potentials of the wave motion. Using the expression for Φ_2 in (2.1), one can ensure the validity of the flow condition on (1.2) accurately to within $(a/h)^2$ (with error $(a/h)^4$), which is the first approximation in Havelock's method.

Substituting expressions (2.1) for $\Phi_{1,2}$ into Eqs.(1.1) and the boundary conditions (1.2)-(1.5), we obtain a boundary-value problem for the unknowns $\varphi_{1,2}$:

$$\Delta\varphi_1 = 0, \ |x| < \infty, \ H \geqslant y > 0 \tag{2.2}$$

$$\begin{gathered}\Delta\varphi_2 = 0, \ |x| < \infty, \ y < 0, \ (x, y) \overline{\in} C \\ [\rho_2\varphi_2^{\cdot\cdot} - \rho_1\varphi_1^{\cdot\cdot} + g(\rho_2 - \rho_1)\varphi'_{1,2y}]_{y=0} = 2\rho_2 a^2 h (x^2 + h^2)^{-1} c^{\cdot\cdot}(t) \\ \varphi_{1y}'(x, 0, t) = \varphi_{2y}'(x, 0, t), \ \varphi_{1y}'(x, H, t) = 0\end{gathered}$$

Here and below, dots denote differentiation with respect to time, and primes together with subscripts denote differentiation with respect to x, y. The initial conditions for the wave motion potentials $\varphi_{1,2}$ are

$$\varphi_{1,2}(x, y, 0) = 0, \ [\rho_1\varphi_1^{\cdot} - \rho_2\varphi_2^{\cdot} - 2\rho_2 a^2 h (x^2 + h^2)^{-1} c^{\cdot}]_{y=0,\, t=0} = 0 \tag{2.3}$$

Using Fourier transforms, we can express the potentials $\varphi_{1,2}$ in the form $(k \geqslant 0$ denotes the wave number)

$$\begin{gathered}\varphi_1(x, y, t) = \int_0^\infty A_1(k, t)\frac{\operatorname{ch} k(H-y)}{\operatorname{ch} kH}\cos kx\, dk \\ \varphi_2(x, y, t) = \int_0^\infty A_2(k, t)\, e^{ky}\cos kx\, dk; \quad A_1(k, t) = -A_2(k, t)\operatorname{cth} kH\end{gathered} \tag{2.4}$$

The relationship between the unkowns A_1 and A_2 follows from the third boundary condition (2.2). Using it, we convert the first condition (2.2) into a closed differential equation (with respect to t) for $A_2(k, t)$;

$$A_2^{\cdot\cdot} + \omega^2(k) A_2 = -2a^2\varkappa(k)\, e^{-kh} c^{\cdot\cdot}(t) \tag{2.5}$$

$$\begin{gathered}\varkappa(k) \equiv \rho_2 \operatorname{th} kH (\rho_1 + \rho_2 \operatorname{th} kH)^{-1}, \ k \geqslant 0 \\ \left(\int_0^\infty e^{-kh}\cos kx\, dk = \frac{h}{h^2 + x^2}, \ h > 0\right)\end{gathered}$$

This equation involves an important characteristic of the oscillatory process: $\omega = \omega(k)$ is the frequency of the internal wave with wave number k:

$$\begin{gathered}\omega^2 = \omega^2(k) = (\rho_2 - \rho_1) gk \operatorname{th} kH (\rho_1 + \rho_2 \operatorname{th} kH)^{-1} \equiv \\ (\rho_2 - \rho_1)\rho_2^{-1} gk\varkappa(k), \ k \geqslant 0\end{gathered} \tag{2.6}$$

It follows from this expression that the phase velocity $\omega(k)/k$ is not the same as the group velocity $d\omega(k)/dk$, $k > 0$. From the physical viewpoint, the internal waves possess dispersion; only in the limit as $kH \to 0$ does the dispersion vanish.

The solution of Eq.(2.5) satisfying the initial conditions (2.3), obtained by twice integrating by parts, can be written as

$$A_2(k, t) = -2a^2\varkappa(k)e^{-kh}\left[c(t) - \omega(k)\int_0^t c(\tau)\sin\omega(k)(t-\tau)\,d\tau\right], \quad A_1 = -A_2 \operatorname{cth} kH \tag{2.7}$$

For the sequel, we need mainly the potential Φ_2; the function Φ_1 is determined by (2.4) and (2.7). On the basis of (2.1), (2.4), and (2.7) we can find an explicit expression for Φ_2, involving rather complicated quadratures:

$$\Phi_2(x, y, t) = a^2c(t)z(x, y) - 2a^2\left[c(t)\int_0^\infty \varkappa(k)e^{k(y-h)}\cos kx\,dk - \int_0^t c(\tau)\int_0^\infty \varkappa(k)\omega(k)e^{k(y-h)}\sin\omega(k)(t-\tau)\cos kx\,dk\,d\tau\right] \tag{2.8}$$

$$\Phi_2|_{y\to-\infty} \to 0, \quad \Phi_2|_{|x|\to\infty} \to 0$$

$$|x| < \infty, \; y < 0, \; (x, y) \overline{\in} C, \; 0 \leqslant t \leqslant T < \infty$$

It follows from (2.8) that the conditions at infinity (1.3) are satisfied. If the motion of the cylinder $s^\cdot(t) = c(t)$ $(c(0) = 0)$ is given, we can use (1.4) and our expressions for the potentials to represent the form of the interface $y = \eta(x, t)$ (here we need the function Φ_1) in the form

$$\eta(x, t) = (\rho_2 - \rho_1)^{-1}g^{-1}[\rho_2\Phi_2^\cdot(x, 0, t) - \rho_1\Phi_1^\cdot(x, 0, t)], \quad |x| < \infty, \; 0 \leqslant t \leqslant T \tag{2.9}$$

The resulting explicit expression (obtained after substitution of the functions $\Phi_{1,2}(x, y, t)$ turns out to be extremely cumbersome. However, in the limit as $H \to \infty$ $(H \gg h)$, which is often the case in practice, the elevation of the interface, expressed in dimensionless variables, can be brought to a form amenable to analysis (see Sect.3).

3. *Derivation of equations of oscillations of the cylinder allowing for reaction of radiated waves.* In order to determine the displacement $s(t)$ of the centre of mass C of the cylinder (or the point O), we must construct a closed equation of motion incorporating the reaction of the liquid. We first calculate the external forces acting on the body. The reaction force **F** acting on the cylinder is found by integrating the projections of the pressure forces around the contour Σ:

$$\mathbf{F} = -b\oint_\Sigma p\mathbf{n}\,dl, \quad \mathbf{n} = (n_x, n_y)^T \tag{3.1}$$

where $p = p(x, y, t)$ is the pressure in the liquid and **n** the outward normal. Using the Bernoulli integral, we obtain

$$p = p(x, y, t) = \rho_2[\Phi_2^\cdot(x, y, t) - gy] \tag{3.2}$$

Evaluating the right-hand side of (3.2) with the help of (2.8), substituting the result into (3.1) and integrating, we obtain $\mathbf{F} = (F_x, F_y)^T$:

$$F_x \equiv 0, \quad F_y = \pi a^2b\rho_2\left[g - c^\cdot(t)\left(1 + \frac{a^2}{4h^2} - 2a^2\int_0^\infty k\varkappa(k)e^{-kh}\,dk\right)\right] - 2\pi a^2b\rho_2\int_0^t c(\tau)\int_0^\infty \varkappa(k)\omega^2(k)e^{-2kh}\cos\omega(k)(t-\tau)\,dk\,d\tau \tag{3.3}$$

The displacement $s(t)$ of the centre of intertia C of the cylinder is conveniently reckoned from the equilibrium position. Using the vertical component of the pressure force as expressed by (3.3), we obtain a closed integrodifferential equation (IDE) for the vertical oscillations of the body on the elastic member near the interface of the two liquids:

$$s^{\cdot\cdot} + \Omega^2 s = -2a^2 \frac{M_2}{M}\int_0^t s^{\cdot}(\tau)\int_0^\infty k\varkappa(k)\,\omega^2(k)\,e^{-2kh}\cos\omega(k)(t-\tau)\,dk\,d\tau \tag{3.4}$$

$$\Omega^2 = \frac{\lambda}{M},\quad M_2 = \pi a^2 b\rho_2,$$
$$M = M_C + M_2\left[1 + \frac{a^2}{4h^2} - 2a^2\int_0^\infty k\varkappa(k)\,e^{-2kh}\,dk\right]$$

Here M is the effective mass of the cylinder (including the added mass of the liquid), M_C is its true mass and Ω is the frequency of oscillations of the cylinder when there is no wave radiation (if $\rho_1 = \rho_2$ or $\omega(k) \equiv 0$). The IDE (3.4) must be solved with initial conditions

$$s(0) = s^0,\ s^{\cdot}(0) = c(0) = 0 \tag{3.5}$$

We have thus obtained as IDCP (3.4), (3.5), whose solution $s(t)$ over the relevant time interval $0 \leqslant t \leqslant T < \infty$ enables all the unknown characteristics of the motion of the system to be found as quadratures (see Sects.2, 3). We now introduce dimensionless parameters and variables:

$$H_* = H/h,\ \ \delta = \rho_1/\rho_2,\ \ \xi = kh,\ \ t_* = \Omega t \tag{3.6}$$
$$\gamma^2 = \frac{g}{\Omega^2 h}(1-\delta) > 0,\ \ \varepsilon = 2\left(\frac{a}{h}\right)^2\frac{M_2}{M}\gamma^2\ \ \ (\varepsilon \ll \gamma^2 \sim 1)$$

In view of the linearity and homogeneity of the equation, the unknown s can be nondimensionalized relative to any convenient quantity, say h (or s^0, in which case $s(0) = 1$). As a result we obtain the following IDCP:

$$s^{\cdot\cdot} + s = -\varepsilon\int_0^{t_*} K(\delta, H_*, \gamma(t_* - \tau))\,s^{\cdot}(\tau)\,d\tau \tag{3.7}$$
$$s(0) = s^0,\ (=1),\ s^{\cdot}(0) = 0$$
$$K(\delta, H_*, \gamma t_*) = \int_0^\infty \frac{\xi^2\zeta}{(\delta+\zeta)^2}\cos\left(\frac{\xi\zeta}{\delta+\zeta}\right)^{1/2}\gamma t_*\,d\xi$$
$$\zeta = \operatorname{th}\xi H_*,\ 0 < \varepsilon \ll 1,\ \delta \geqslant 0,\ H_* > 0,\ \gamma > 0,\ 0 \leqslant t_* \leqslant T_*(\varepsilon)$$

For applications, it is of interest to investigate the solutions of problem (3.7) over the asymptotically large interval of the argument $t_* \in [0, T_*(\varepsilon)]$, where $T_*(\varepsilon) \to \infty$ as $\varepsilon \to 0$, in which the behaviour of the system changes in a qualitatively significant way. A complete investigation of the motion as described by the IDCP (3.7) is difficult, as it involves a large number of parameters: $\varepsilon, \gamma, \delta, H_*$. We therefore consider the limiting case $H_* \to \infty$ (an infinitely deep upper layer); computations show that the solutions at $H_* = \infty$ and $H_* \simeq 1$ (i.e., $H \simeq h$) are practically identical. Problem (3.7) involves the parameters ε, γ and can be reduced to the following form (for convenience we simply put $t_* = t$):

$$s^{\cdot\cdot} + s = -\varepsilon\int_0^t K(\gamma(t-\tau))\,s^{\cdot}(\tau)\,d\tau,\quad s(0) = s^0(=1),\ s^{\cdot}(0) = 0 \tag{3.8}$$
$$K(\gamma t) = \int_0^\infty \xi^2 e^{-2\xi}\cos\sqrt{\xi}\,\gamma t\,d\xi,\ \ \gamma^2 = \frac{g^*}{\Omega^2 h},\ \ g^* = g\frac{1-\delta}{1+\delta}$$
$$\varepsilon = \left(\frac{a}{h}\right)^2\frac{M_2}{M}\gamma^2 \ll 1,\ \ \gamma \sim 1;\ \ 0 \leqslant t \leqslant T(\varepsilon),\ \ T(\varepsilon)|_{\varepsilon\to 0} \to \infty$$

Attention is now given to constructing a solution $s = s(t, \gamma, \varepsilon)$ of the IDCP (3.8). As seen in Sect.2, once this solution has been found one can determine the velocity potentials

$\Phi_{1,2}(x, y, t)$ in both liquids, the velocity fields $\mathbf{v}_{1,2} = -\nabla\Phi_{1,2}$ and pressures $p_{1,2}$ (Sects. 2,3), as well as the interface $y = \eta(x, t)$ (Sect.2). In particular,

$$\eta_*(x_*, t) = \int_0^t s^{\cdot}(\tau) \int_0^\infty \xi e^{-\xi} \cos \xi x_* \cos \sqrt{\xi}\, \gamma (t - \tau)\, d\xi\, d\tau \tag{3.9}$$

$$x_* = \frac{x}{h}, \quad \eta_*(x_*, t) = \frac{\eta(x_* h, t)}{\mu h}, \quad \mu = 2\left(\frac{a}{h}\right)^2 \frac{1}{1+\delta} \ll 1$$

A formal solution of the IDE (3.8), which has a convolution-type kernel, can be found by operator methods /7/, as done, e.g., in /1, 2/. However, it is generally impossible to analyse such solutions, because of the complicated analytical structure of the kernel $K(\gamma t)$ and its transforms. To construct the required solution and interface of the liquids over an asymptotically large time interval, numerical methods are available. Below (Sect.5) we shall present the results of computations and an analysis of the motion of the system for various values of the parameters ε, γ. In connection with the global characteristics of linear oscillations of system (3.8) it is also important, from both theoretical and applied standpoints, to investigate the steady-state forced oscillations induced by a harmonic external force on the body.

4. Forced oscillations of the oscillator near the liquid interface. The motion of the body under an impressed harmonic force, in non-dimensional variables, is described by the following IDCP:

$$S^{\cdot\cdot} + S = -\varepsilon \int_0^t K(\gamma(t-\tau))\, S^{\cdot}(\tau)\, d\tau + f_0 \cos \alpha t \tag{4.1}$$

$$S(0) = S^{\cdot}(0) = 0, \; \alpha \in [0, \infty) \; (f_0 = 1)$$

By using Laplace transforms /7/, we obtain expressions for the transforms of the solution and the kernel:

$$S^*(p, \alpha, \gamma, \varepsilon) = p/R^*, \; R^* = (p^2 + \alpha^2)[1 + p^2 + \varepsilon p K^*(p, \gamma)] \tag{4.2}$$

$$K^*(p, \gamma) = \int_0^\infty \frac{p\xi^2 e^{-2\xi}\, d\xi}{p^2 + \xi\gamma^2} = \frac{p}{4\gamma^2} - \frac{p^3}{2\gamma^4} + \frac{p^5}{\gamma^4} \int_0^\infty \frac{e^{-2\xi}\, d\xi}{p^2 + \xi\gamma^2}$$

$$p = \sigma + i\nu, \quad \sigma \geqslant \sigma_0 \geqslant 0, \quad |\nu| < \infty$$

Inverting the transform $S^*(p)$ (4.2) produces a standard expression for $S(t)$ — the solution of problem (4.1):

$$S = S(t, \alpha, \gamma, \varepsilon) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{p e^{pt}\, dp}{R^*(p^2, \alpha, \gamma, \varepsilon)} \tag{4.3}$$

Periodic steady-state motion is represented here by the contributions to the integral (4.3) from the imaginary poles $p = \pm i\alpha$. However, evaluation of the residues at these poles yields a divergent integral in the expression for $K^*(\pm i\alpha, \gamma)$ (which is convergent only in the sense of the principal value). To avoid this complication, we transform (4.3) by making the substitution $p = \sigma + i\nu$, $-\infty < \nu < +\infty$ ($\sigma = \text{const}$) and evaluate the residues at $p = \pm i\alpha$. The required amplitude-frequency characteristic (AFC) is

$$A = A(\alpha, \gamma, \varepsilon) = 1/R(\alpha^2, \gamma^2, \varepsilon) \tag{4.4}$$

$$R(\alpha^2, \gamma^2, \varepsilon) = \left| 1 - \alpha^2 - \frac{\varepsilon}{4} \frac{\alpha^2}{\gamma^2} \left(1 + 2\frac{\alpha^2}{\gamma^2}\right) - \varepsilon \frac{\alpha^6}{\gamma^4} Q(\alpha^2, \gamma^2) \right| =$$

$$\{[1 - \alpha^2 - \varepsilon f(\chi)]^2 + \varepsilon^2 g(\chi)\}^{1/2} \equiv X(\alpha^2, \chi, \varepsilon)$$

$$Q(\alpha^2, \gamma^2) = \lim_{\sigma \to 0} \int_0^\infty \frac{e^{-2\xi}\, d\xi}{(\sigma + i\alpha)^2 + \xi\gamma^2} = \frac{1}{\gamma^2} I(\chi) - i \frac{2\pi}{\gamma^2} e^{-2\chi}$$

$$I(\chi) = \int_0^\infty \frac{\xi \cos 2\xi - \chi \sin 2\xi}{\xi^2 + \chi^2}\, d\xi, \quad \chi = \frac{\alpha^2}{\gamma^2}, \quad 0 \leqslant \chi < \infty$$

$$f(\chi) \equiv (\chi/4)(1 + 2\chi) + \chi^3 I(\chi), \; g(\chi) \equiv (2\pi)^2 \chi^6 e^{-4\chi}$$

Analysis of the function $R(\alpha^2, \gamma^2, \varepsilon)$ reveals that at fixed but fairly small $\varepsilon > 0$, it has a minimum as a function of α^2 and a maximum as a function of γ^2 (a saddle point); accordingly, the function X has a minimum with respect to α^2 and a maximum with respect to χ. The amplitude at this point has a maximum as a function of α^2, which tends to infinity like $1/\varepsilon$ as $\varepsilon \to 0$, and a minimum with respect to γ^2 (or χ). Equating the derivatives of X^2 with respect to α^2 and χ to zero, we obtain equations for the minimax point (α_0, γ_0):

$$\begin{gathered}\partial X^2/\partial\alpha^2 = -2\,[1 - \alpha^2 - \varepsilon f(\chi)] = 0, \ \alpha^2 = 1 - \varepsilon f(\chi) \\ \partial X^2/\partial\chi = -2\varepsilon f'(\chi)\,[1 - \alpha^2 - \varepsilon f(\chi)] + \varepsilon^2 g'(\chi) = 0, \ g'(\chi) = 0\end{gathered} \tag{4.5}$$

Differentiation of $g(\chi)$ enables one to determine the required value $\chi_0 = {}^3/_2$. Substitution of this value into expression (4.5) for α^2 yields the required values of both variables $\alpha_0(\varepsilon)$, $\gamma_0(\varepsilon)$ and the minimax value $A_0(\varepsilon)$:

$$\begin{gathered}\alpha_0^2 = \alpha_0^2(\varepsilon) = 1 - \varepsilon f({}^3/_2), \quad f({}^3/_2) \simeq -0.186 \\ \gamma_0^2 = \gamma_0^2(\varepsilon) = {}^2/_3\,[1 - \varepsilon f({}^3/_2)] \\ A_0 = A_0(\varepsilon) = A(\alpha_0(\varepsilon), \gamma_0(\varepsilon), \varepsilon) = \\ \varepsilon^{-1} g^{-1/2}({}^3/_2) = \varepsilon^{-1} 4e^3/(27\pi) \simeq 0.94\varepsilon^{-1}\end{gathered} \tag{4.6}$$

For fixed values of the parameters ε, γ, the value of α^* corresponding to the maximum of A is found by solving the equation $dX^2/d\alpha^2 = 0$ (say, by Picard's method), i.e.,

$$\begin{gathered}\alpha^2 = 1 - \varepsilon f(\chi) + \varepsilon\gamma^{-2} f'(\chi)\,[1 - \alpha^2 - \varepsilon f(\chi)] - \varepsilon^2\gamma^{-2} g'(\chi) \\ \alpha^{*2} = \alpha^{*2}(\gamma, \varepsilon) = 1 + O(\varepsilon) = 1 - \varepsilon f(\gamma^{-2}) + O(\varepsilon^2) = \\ 1 - \varepsilon f(\gamma^{-2}) + \varepsilon^2\gamma^{-2} f'(\gamma^{-2}) f(\gamma^{-2}) - {}^1/_2\varepsilon^2\gamma^{-2} g'(\gamma^{-2}) + O(\varepsilon^3) \equiv \\ \max_\alpha A(\alpha, \gamma, \varepsilon) = A(\alpha^*(\gamma, \varepsilon), \gamma, \varepsilon) = A_0(\varepsilon) + O(\varepsilon), \ \gamma \sim 1\end{gathered} \tag{4.7}$$

It follows from (4.6) that the resonance frequency $\alpha_0(\varepsilon)$ is greater than unity, i.e., than the "natural frequency", by an amount $O(\varepsilon)$: $\alpha_0 \simeq 1 + 0.093\varepsilon$; this property remains valid for $\gamma \sim 1$ close to $\gamma_0 = \gamma_0(\varepsilon)$. It follows from (4.6) that for fixed, sufficiently small $\varepsilon > 0$, the AFC shows the standard shape of a unimodal curve (with a single maximum as a function of α). The maximum of $A = A(\alpha, \gamma, \varepsilon)$ is reached at $\alpha^* \simeq 1 - {}^1/_2\varepsilon f(\gamma^{-2})$ and it is equal to $A^* = A^*(\gamma, \varepsilon) = A_0(\varepsilon) + O(\varepsilon)$ (see (4.7)), i.e., it is independent, to within a relative error $O(\varepsilon^2)$, of γ, $\gamma \sim 1$. If there is a significant change in the value of γ, then in the limiting cases $\varepsilon\gamma^{-6} \to 0$ and $\gamma \to \infty$ the resonance frequencies are $\alpha^* \simeq 1 - \varepsilon^2\gamma^{-2} g'(\gamma^{-2})$ and $\alpha^* \simeq 1 + \varepsilon / (8\gamma^2)$, respectively, and in the limit $\alpha^* = 1$ ($\gamma = 0$, $\varepsilon\gamma^{-6} = 0$, $\gamma = \infty$). It should be noted that in the limiting cases system (4.1) is actually a differential equation without an integral term (i.e., without damping), of the form $S^{\cdot\cdot} + S = \cos \alpha t$, for which $A(\alpha, \gamma, \varepsilon) \to \infty$ as $\alpha \to 1$. The normal modes of the initial, conservative system (3.8) will be undamped and harmonic: $s(t) = \mathrm{const}$, $t \geqslant 0$. Calculation of the oscillations of the cylinder according to the IDCP (3.8), the wave motions of the liquid at the interface according to formula (3.9) and the ARC (4.4) for arbitrary values of $\varepsilon\,(0 < \varepsilon \lesssim 1)$, $\gamma\,(0 < \gamma < \infty)$ presupposes the development of numerical algorithms.

5. ***Results of computations and qualitative conclusions.*** Computations using formula (4.4) (Fig.2) have established that for fixed, sufficiently small $\varepsilon > 0$ ($\varepsilon = 0.1$) the resonance frequency $\alpha^*(\gamma, \varepsilon)$ increases as γ increases ($\gamma \geqslant 0.1$), goes through the value $\alpha^*(\gamma, \varepsilon) = 1$, reaches a maximum and then decreases, approaching unity - the natural frequency. Thus, for all $\gamma \sim 1$ the variations in the resonance frequency are of the order of ε, $\varepsilon \ll 1$. The maximum amplitudes $A^* = A(\alpha^*(\gamma, \varepsilon), \gamma, \varepsilon)$ agree with formulae (4.6), (4.7) and the analysis in Sect.4. One observes a very high selective capacity of the oscillating system in relation to the frequency α for sufficiently small $\varepsilon > 0$ and a marked dependence on the parameter γ near $\gamma \simeq \gamma_0(\varepsilon)$. At $\gamma = \gamma_0(\varepsilon)$ the amplitude is a minimum as a function of γ and a maximum as a function of α, indicating maximum damping of the oscillations, i.e., maximum interaction of oscillator and liquid.

An increase in ε leads to a drop in the maximum amplitude according to (4.6), (4.7), i.e., to increased generation of internal waves and damping of oscillations in the oscillator (Fig.2). At $\varepsilon \geqslant 0.35$ one observes a point of deflection to the left of the maximum, which is deformed when ε increases further so that the AFC has two local maxima and a minimum in the neighbourhood of $\alpha = 1$. This behaviour of the AFC is typical of two-frequency linear oscillating systems with significant damping.

Analysis of the AFC makes it possible to determine the global properties of the oscillating system described by the IDCP (3.8). A program has been worked out to solve this problem, involving a laborious procedure for constructing the kernel $K(\theta)$ of the integral operator with high accuracy over a long interval of time t $(\theta = \gamma t)$. The displacements $s(t, \gamma, \varepsilon)$ were computed over a time interval in the course of which there was a significant drop in the amplitude of the oscillations and considerable generation of internal waves. After constructing the solution of system (3.8) by formula (3.9), the space-time picture of the variations in the interface $\eta_*(x_*, t, \gamma)$ was computed. To make the results more readily visualizable, the value of ε was taken to be fairly large $(\varepsilon = 0.5)$, while those of γ were taken both near the maximum point $\gamma_0(\varepsilon)$ $(\gamma \simeq \gamma^0 = \sqrt{2/3})$ and far from it $(\gamma = 0.5)$.

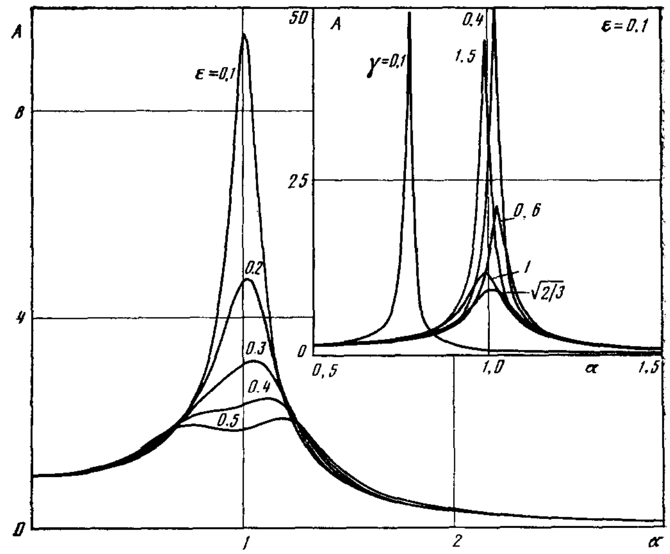

Fig.2

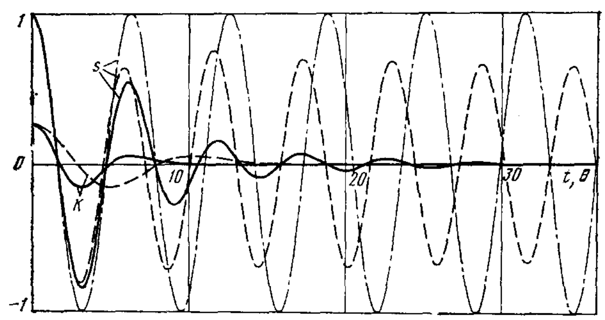

Fig.3

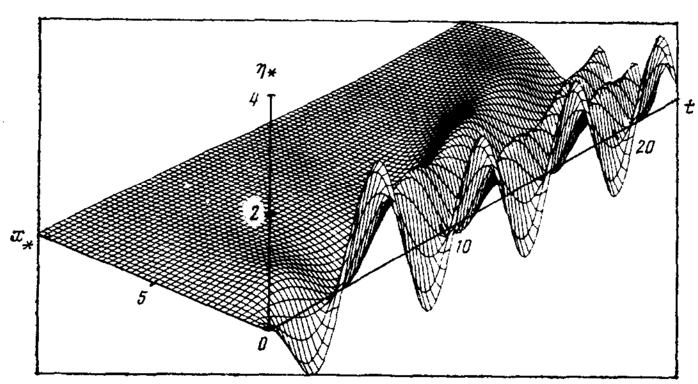

Fig.4

Fig.3 presents the results of computations of $K(\theta)$ and $s(t, \gamma, \varepsilon)$ at $\gamma = \gamma^0$ (the solid curves) and $\gamma = 0.5$ (the dashed curves); the cosinusoid $s_0 = s(t, \gamma, 0) = \cos t$ (the dash-dot curve) describes the oscillations of the oscillator at $\varepsilon = 0$. From the shape of the curves for different γ one can conclude that the oscillations may differ significantly in nature. At $\gamma \simeq \gamma^0$ the oscillations of the cylinder are damped far more rapidly than at $\gamma = 0.5$, in agreement with our analysis of the AFC.

Furthermore, comparison of the space-time pictures of the interface at $\varepsilon = 0.5$ and $\gamma = 0.5$ (Fig.4), $\gamma = \gamma^0$ (Fig.5) shows that the case $\gamma = 0.5$ corresponds to weak interaction between the oscillator and liquid. The excitation of internal waves at points of the surface with large x_* values is negligible and the damping of the oscillations is slow. At $\gamma = \gamma^0$ $(\gamma \simeq \gamma_0)$ the oscillator-medium interaction is considerably stronger: the oscillations are damped rapidly. Internal waves are generated and propagate over the interface.

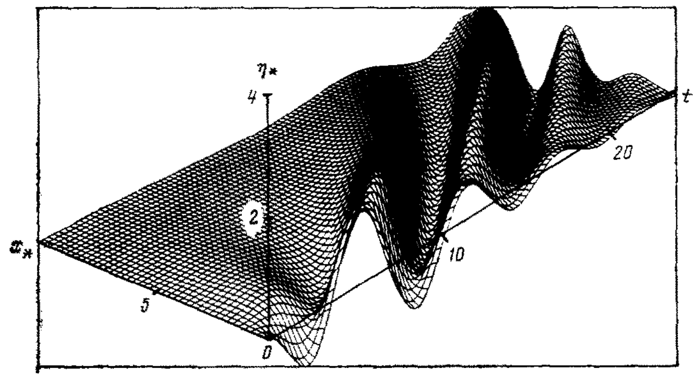

Fig.5

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